Surfski Science
1. The Physics of Surfing a Wave

David Glover
Cambridge Canoe Club, UK
davidmglover@btinternet.com
December 2017

INTRODUCTION

In a previous paper I reviewed the forces acting on a K1 when paddled on flat water.
https://www.academia.edu/32163425/Kayak_performance_measurements_with_a_Kayak_Power_Meter.pdf

As a recent convert to surfski paddling, I've now included the action of waves. This paper reports an analysis of the forces on a surfski as the paddler surfs, or attempts to surf an ocean wave. The wave conditions necessary to produce sustained, effortless 'paddle-in-the-air' runs are calculated, and a prediction of the maximum possible speed of a surfski is made.

The novice surfski paddler soon discovers that not all waves are alike. In deep water big, fast moving waves can help an experienced paddler accelerate the ski to 20-30 km/h for short runs, but the wave crests are travelling faster than this. A crest soon moves ahead of the boat, the boat speed drops, and the paddler must catch the next wave to boost his or her speed again.

In shallower water (depth less than 5 meters say), the wave crests travel more slowly. In these conditions even a relatively modest wave, height 0.5 m or less, can potentially raise the stern and carry the boat along at the wave speed, with no effort from the paddler.

What are the factors that determine which waves can be surfed effortlessly and which cannot? The charts presented in the Part 1 of this post predict when surfing is possible from three readily available parameters that, together, determine the speed and 'surfability' of a typical wave on a particular day. The three parameters are the wave height, H, the wave period, T, and the water depth, d.

The details of the theory used to construct the charts are set out in Part 2 of the paper.

PART 1
SURFABLE OR NOT?

1.1 Sea conditions

My regular paddle starts from Avon Beach on Christchurch Bay in Dorset UK. Before I paddle I check the predicted sea state on the website Big Salty:
https://bigsalty.com/en/weather/forecast/71/

This gives me the wind speed and direction, swell height, H, and swell period, T (equivalent data are available from surfing and boardsailing websites for paddling locations around the World).

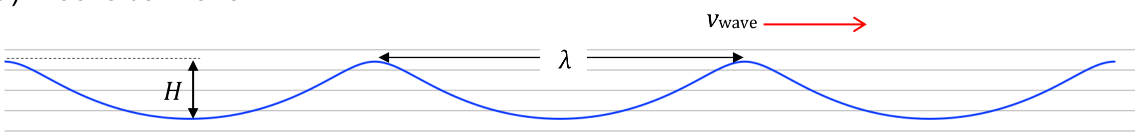
The water depth, *d*, can be found from an online marine chart (for example https://webapp.navionics.com/?lang=en#boating@10&key=amntHjcbI)

The average depth in Christchurch Bay is 5-6 m, increasing to 20 m offshore. On the western side of the bay there is a shallow area of sandbanks 1-3 m depth. This area produces good waves for my novice standard surfing.

1.2 Wave characteristics
Figure 1(a) shows an idealised ocean wave. The wave profile is that of the trochoidal or Gerstner[1] wave. This wave shape corresponds more closely to that of a real ocean wave than does the profile of the familiar sinusoidal waveform in Figure 1(b). The peaks of the trochoidal wave are steeper and troughs more level than the peaks and troughs of a sine wave. The equations of the trochoidal waveform are introduced in Part 2.

Figure 1
Wave shapes

(a) Trochoidal wave



(b) Sinusoidal wave

The parameters that describe the wave are as follows:

Wave height H
The wave height H is the height of the wave peak (in metres) above the trough.

Wavelength λ
The wavelength λ is the separation of two adjacent wave crests.

Wave period T
The period T is the time (in seconds) between consecutive wave crests passing a fixed point. If you are standing on the shore, it is the time between one crest and the next arriving on the beach.

Wave speed v_{wave}
The wave speed v_{wave} is the distance travelled by the wave per second, and is thus given by:

$$v_{\text{wave}} = \frac{\lambda}{T}$$

and so,

$$v_{\text{wave}}\, T = \lambda$$

Wave steepness S

The wave steepness S is defined as π times the ratio of the wave height to the wavelength,

$$S = \frac{\pi H}{\lambda}$$

(Wave steepness is sometimes defined without the factor π as the simple ratio $\frac{H}{\lambda}$; however the inclusion of π at this stage simplifies later equations involving S.)

Waves with the same steepness have the same shape, independent of their wavelength. For example, a wave with height 2 m and wavelength 20 m has the same steepness as a 4 m high wave with wavelength 40 m ($S = 0.314$).

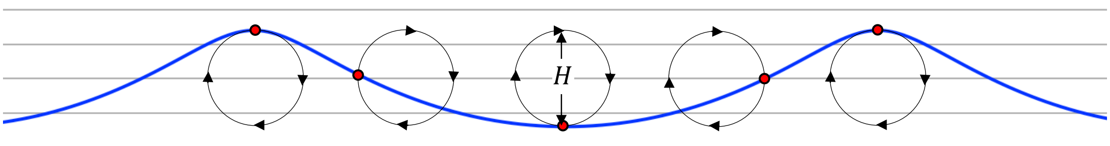
Limit of steepness

Michell (2) used the wave theory of Stokes (3) to predict the maximum steepness of an ocean wave; he calculated this to be 0.44 (this is the steepness of the waveform shown in Figure 1(a)). For $S > 0.44$ the crest becomes unstable and 'breaks'. Observations of deep-water waves confirm a figure of around 0.44 as the limiting steepness for the rear wave face, whilst the absolute limit for the front face steepness is higher at 0.55 – though this extreme steepness was observed in only 1 in 100000 waves (4).

Particle motion

A fundamental characteristic of a travelling wave is that the particles of the medium do not travel along with the wave; rather they oscillate about fixed points with the same period as the wave motion. In a transverse mechanical wave the direction of the particle motion is perpendicular to the direction of wave propagation; for a longitudinal wave (sound for example) the particles oscillate along the direction of propagation. The motion of a surface particle in a water wave has both longitudinal and transverse components, which combine to produce circular orbits. The diameter of the orbits is equal to the wave height H.

Fig 1.2
Particle motion in a water surface wave



Circular motion of the particles produces the trochoidal wave profile.

Dispersion

Ocean waves are dispersive. This means that their speed varies with their wavelength, the longer the wavelength the greater the wave speed. The wave speed is also a function of the water depth, d; as a wave travels from deep water into shallower water its period T is constant, but its speed and wavelength decrease – the wave crests slow and 'bunch together'. The relationship between the wave speed, the wavelength and the water depth is given by the dispersion equation for water surface waves:

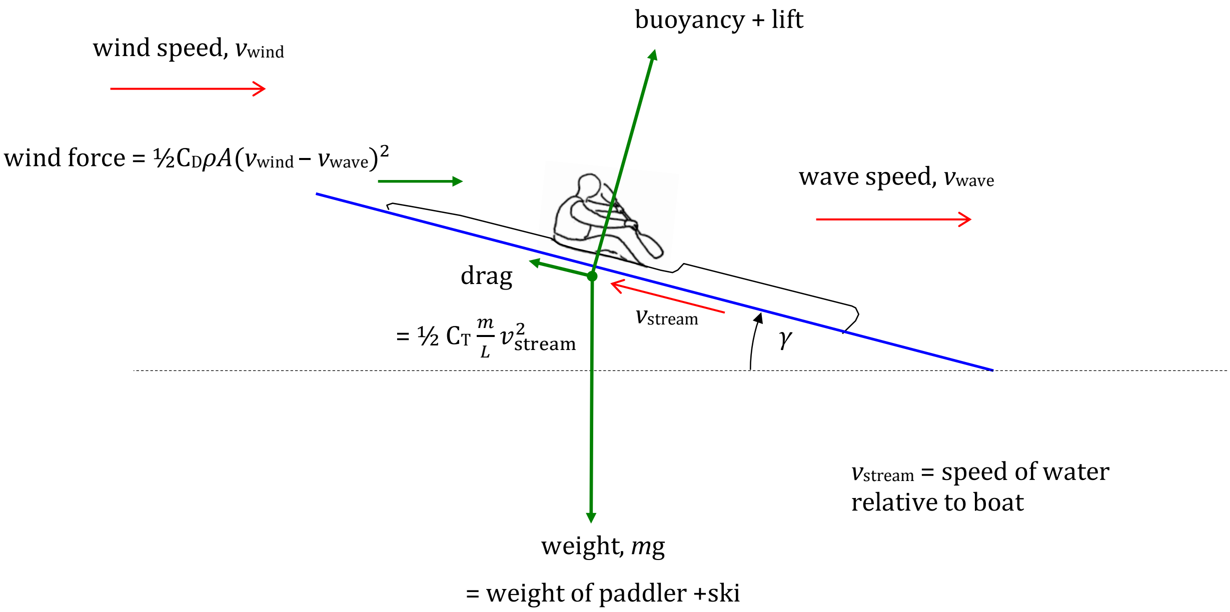
$$v_{\text{wave}} = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(2\pi\frac{d}{\lambda}\right)}$$

1.3 Surfability of a wave

Figure 1.3 shows the forces acting on a surfski when the paddler is surfing without effort in the direction perpendicular to the line of the wave crest.

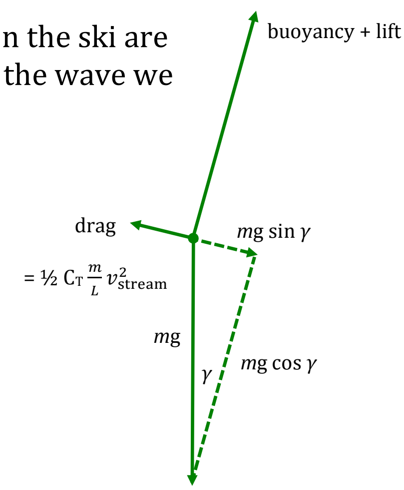
The angle γ is the angle of the slope of the wave surface at the point of surfing. The ski is travelling horizontally at the wave speed v_{wave} ; its position relative to the wave profile is constant, and so there is no vertical motion.

Figure 1.3
Forces when surfing a wave effortlessly at constant speed



To simplify the calculation we consider downwind paddling with the wind speed equal to the wave speed. In this case the wind force on the paddler (½$C_D\rho A(v_{\text{wind}} - v_{\text{wave}})^2$) drops to zero as the ski and wave speeds match. This a realistic situation, corresponding to the condition of a 'fully developed sea.'

Since the speed of the surfing ski is constant, the forces on the ski are balanced in every direction. Resolving down the slope of the wave we have:



$$mg\sin\gamma - ½\, C_T \frac{m}{L} v_{\text{stream}}^2 = 0$$

or,

$$mg\sin\gamma = ½\, C_T \frac{m}{L} v_{\text{stream}}^2$$

Here $mg \sin \gamma$ is the component of the weight (ski plus paddler) acting down the slope. This balances the drag force, ½ $C_T \frac{m}{L} v_{\text{stream}}^2$, which acts in the opposite direction.

The gravitational force and the drag force are both proportional to the mass and so m cancels on either side of the equation to give

$$\text{g} \sin \gamma = \frac{C_T}{2L} v_{\text{stream}}^2 \qquad (1)$$

In this equation C_T is the total drag coefficient of the hull, L the boat length and v_{stream} the speed of the water relative to the boat. It is important to note that v_{stream} is not equal to $-v_{\text{wave}}$. This is because the water particles are in motion in their circular orbits. In part 2 we show how v_{stream} is calculated as the vector sum of $-v_{\text{wave}}$ and the water particle velocity v_{p}, which can in turn be expressed in terms of v_{wave} and the wave steepness S.

Equation (1) is the basis for establishing if a wave will sustain surfing. A surfability condition may be stated as:

$$\sin \gamma \geq \frac{C_T}{2\text{g}L} v_{\text{stream}}^2 \qquad (2)$$

Effortless surfing is possible when $\sin \gamma \geq \frac{C_T}{2\text{g}L} v_{\text{stream}}^2$. If $\sin \gamma$ is less than $\frac{C_T}{2\text{g}L} v_{\text{stream}}^2$, the drag force will decelerate the boat, which will move backwards relative to the wave until either γ has increased sufficiently for surfing to occur, or the wave crest passes. When $\sin \gamma$ is greater than $\frac{C_T}{2\text{g}L} v_{\text{stream}}^2$ the boat will move down the wave until a point of surfing is reached at a lower value of γ.

1.4 Practical surfability charts

In part 2 we show how v_{stream} and $\sin \gamma$ can be expressed in terms of the wave speed and steepness, which in turn can be found from the wave period, water depth and wave height, to produce the practical surfing charts presented below.

For a trochoidal wave the surfabilty condition (2) above is shown to be equivalent to:

$$v_{\text{wave}} \leq \sqrt{\frac{2\text{g}L}{C_T}} \sqrt{\frac{S \sin \theta}{(1 - 2S \cos \theta + S^2)^{3/2}}}$$

This condition states that in order for a wave to be surfable at a given point along its length, the wave speed v_{wave} must be equal to or less than the quantity on the right.

This quantity is the product of two terms: $\sqrt{\frac{2\text{g}L}{C_T}}$ and $\sqrt{\frac{S \sin \theta}{(1-2S \cos \theta + S^2)^{3/2}}}$.

The first term $\sqrt{\frac{2gL}{C_T}}$ is determined by the force of gravity and two characteristics of the ski: its length L and its drag coefficient C_T. This term is proportional to the square root of the ratio L/C_T, confirming that long skis with low drag hulls can potentially surf on higher speed waves than can shorter, higher drag craft. It's interesting to note that weight does not affect the skis surfing capability (weight will however affect the paddler's ability to accelerate the ski to catch a wave, or to maintain top speed when not surfing effortlessly).

The second term is determined by the wave steepness *S,* and by the shape of the wave. For the trochoidal wave shape, to which the expression $\sqrt{\frac{S\sin\theta}{(1-2S\cos\theta+S^2)^{3/2}}}$ applies, θ is a parameter that gives the position of the boat along the wave profile.

Intuitively we might think that the optimum point to surf will be the steepest part of the wave (maximum γ), however - because of the variation of the water particle velocity along the wave profile - the optimum surfing point is displaced backwards towards the wave crest - the steeper the wave, the nearer the sweet spot is to the crest.

By finding the value of θ that maximises the expression $\sqrt{\frac{S\sin\theta}{(1-2S\cos\theta+S^2)^{3/2}}}$ for different values of S, we can use the surfability condition to construct a chart showing if a wave of a given speed v_{wave} (y-axis) and steepness S (x-axis) is surfable or not. This is Chart 1, which is shown on the next page.

The chart area is divided into two regions by the lines

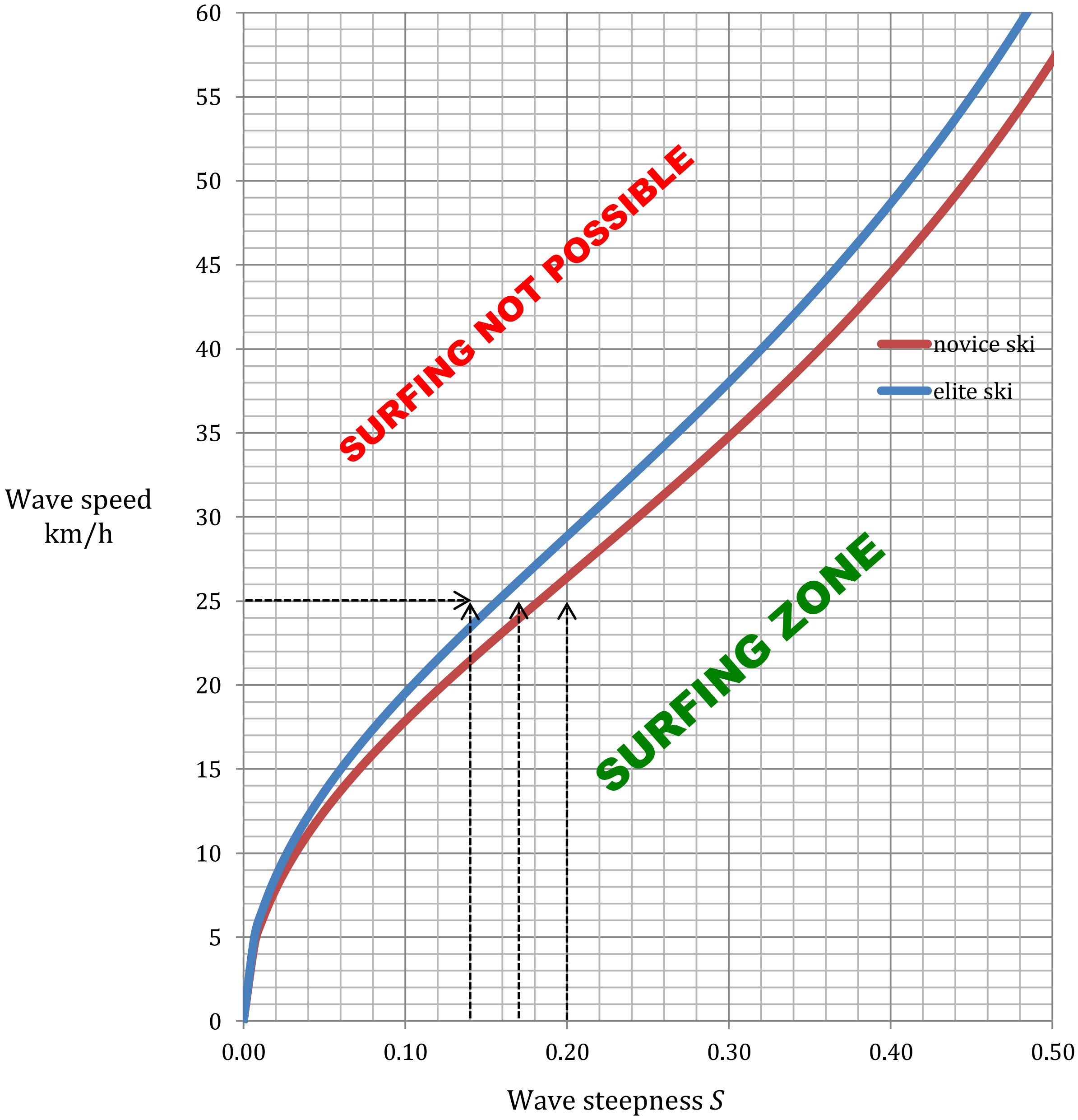
$$v_{\text{wave}} = \sqrt{\frac{2gL}{C_T}}\ \sqrt{\frac{S\sin\theta_{max}}{(1-2S\cos\theta_{max}+S^2)^{3/2}}}$$

calculated for two different skis.

The blue line is drawn for an elite surfski with C_T = 0.44 and L = 6.4 m. The red line is a novice ski with C_T = 0.45 and L = 5.48 m.

Waves with a combination of speed and steepness above the relevant line do not sustain effortless surfing of that ski; waves with a speed and steepness combination below the line can produce continuous paddle-in-the-air surfing fun!

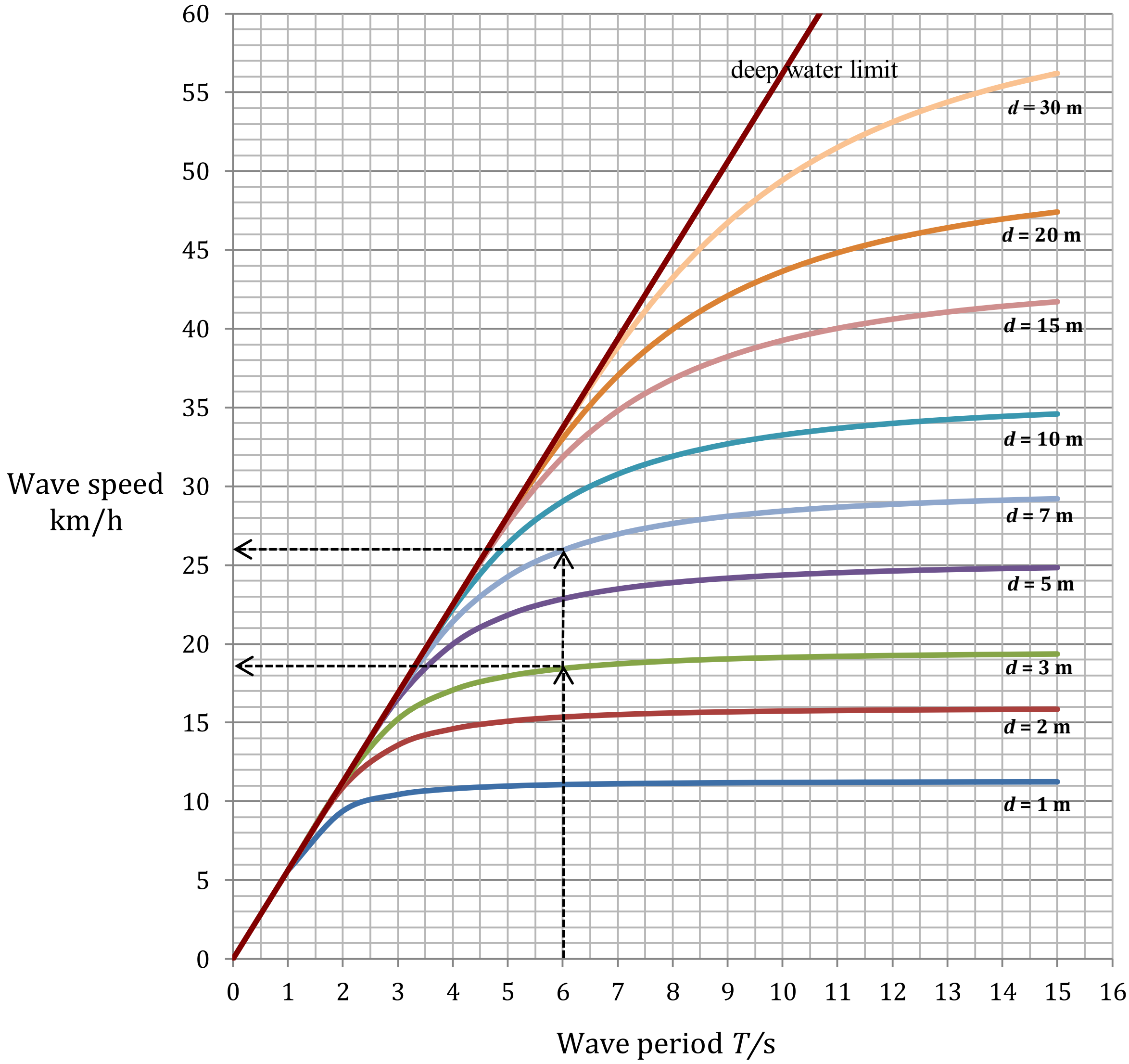
Chart 1
Surfability of waves with speed v_{wave} and steepness S



To use this chart you simply check to see if the point corresponding to a wave of a given speed and steepness lies inside or outside the surfing zone. For example, a wave travelling at 25 km/h with steepness 0.14 cannot be surfed effortlessly, but is surfable by an elite ski if the steepness is 0.17, or by a novice ski when the steepness is 0.20.

Of course, to use the chart you need to know the expected wave speed and steepness on the day you are planning to paddle. These quantities can be found from the predicted wave period, height and the known water depth, with charts 2, 3 and 4 below. But don't worry, you needn't refer to all four charts every time - the charts can be combined in pairs, or as a single 'master chart'. A printable full-page version of the master chart is included at the end of this paper.

Chart 2
Wave speed against period for different water depths *d*



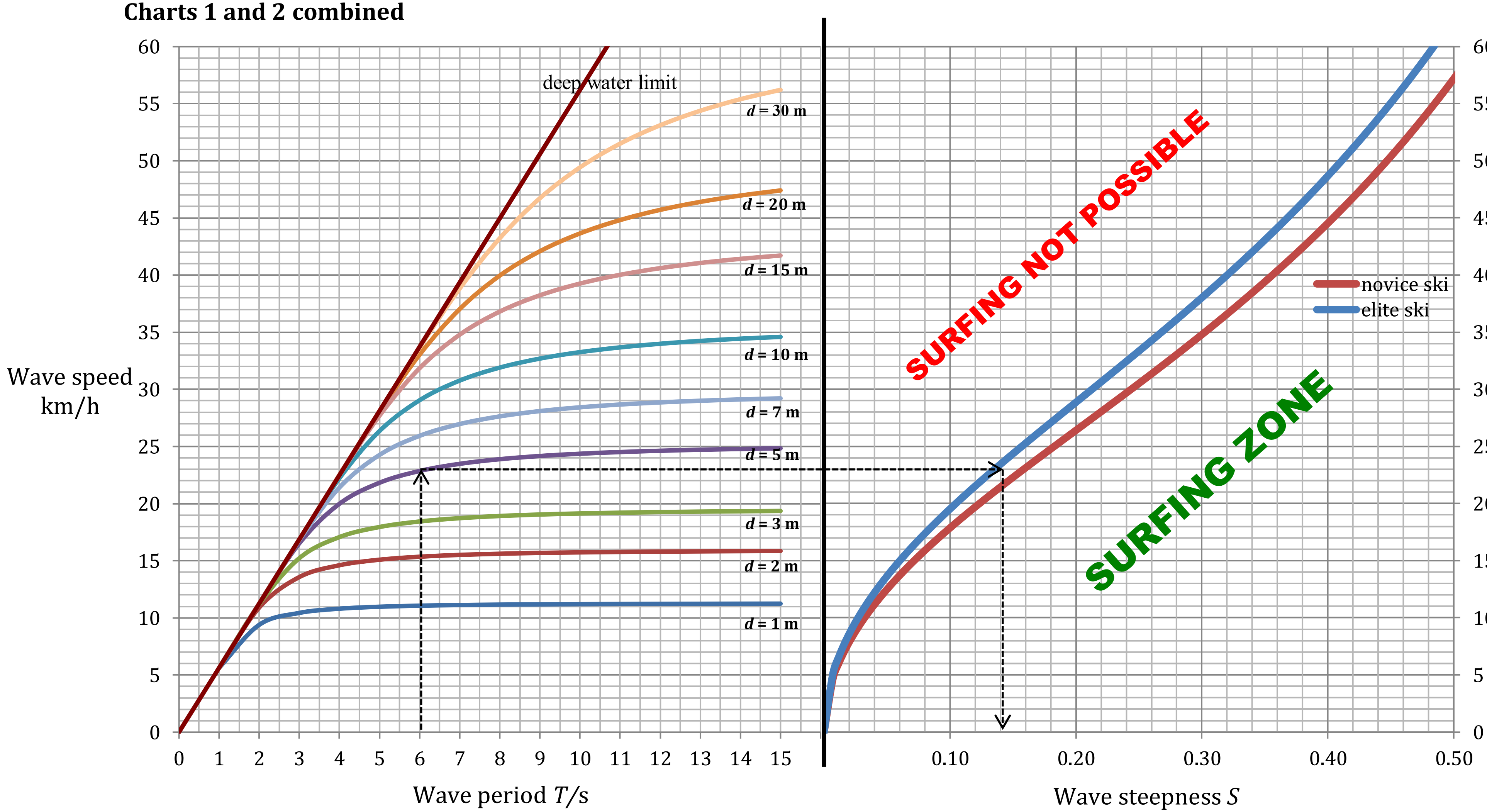
This chart gives us the wave speed from the wave period T and water depth d.

To use this chart draw a vertical line up from the forecast wave period until it meets the curve corresponding to the depth d at your paddling location, then read the wave speed from the y axis. For example, if the wave period is 6 s and the water depth 7 m, then the wave speed will be 26 km/h. If the depth decreases to 3 m the wave speed will decrease to 18.5 km/h.

The coloured lines on this chart have been plotted by eliminating λ between the wave equation, $v = \frac{\lambda}{T}$, and the dispersion relation, $v = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(2\pi\frac{d}{\lambda}\right)}$.

Charts 1 and 2 can be combined to predict if waves of period T, in water of depth d, are potentially surfable or not, and at what speed.

Charts 1 and 2 combined

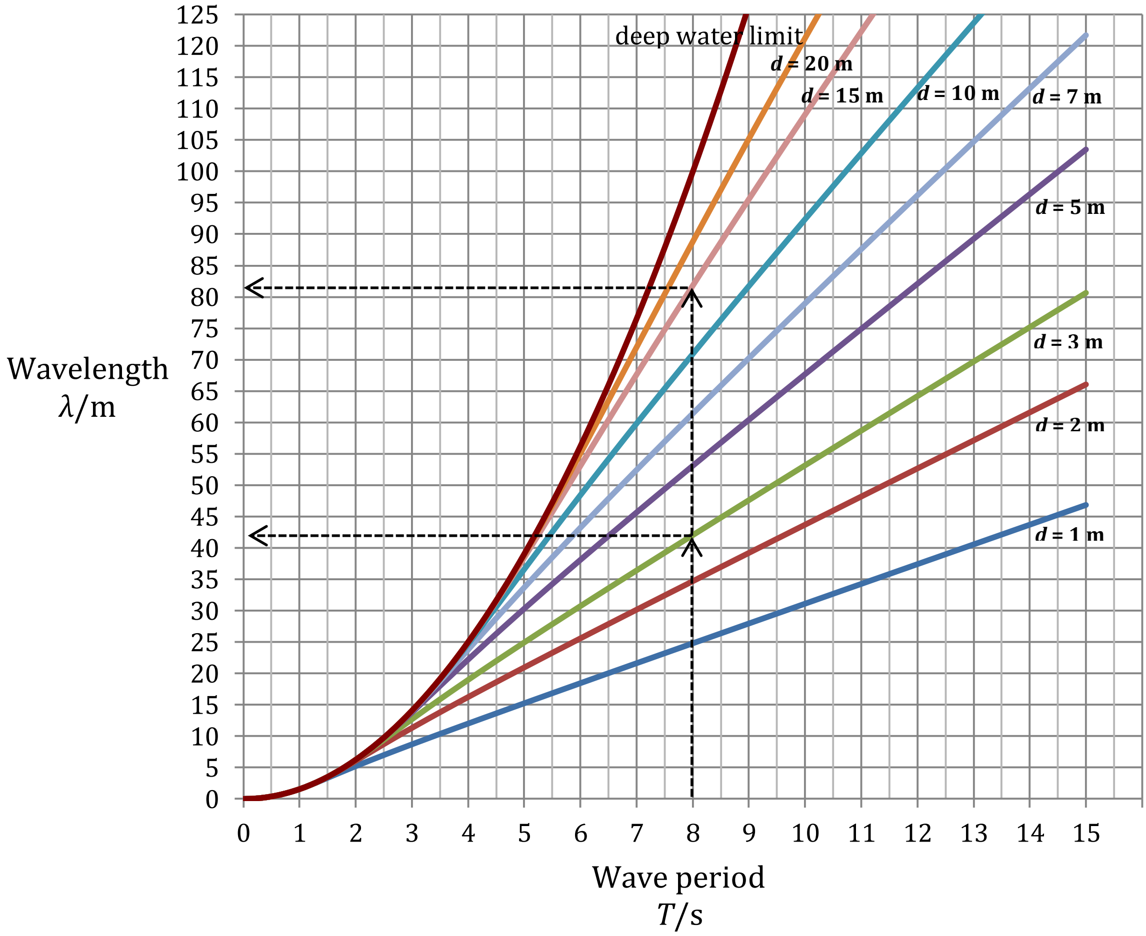


For example, with T = 6 s and d = 5 m, the wave speed will be 23 km/h; surfing is possible for an elite ski if wave steepness S > 0.14.

The final piece in the jigsaw is to use the forecast wave height *H,* together with the period *T* and depth *d*, to predict the expected wave steepness *S*.

As an intermediate step we must first find the wavelength for the given period and depth. This is done with chart 3, which is constructed in a similar way to chart 2, but this time by eliminating v_{wave} between the wave equation and the dispersion equation.

Chart 3
Wavelength against period for different water depths *d*



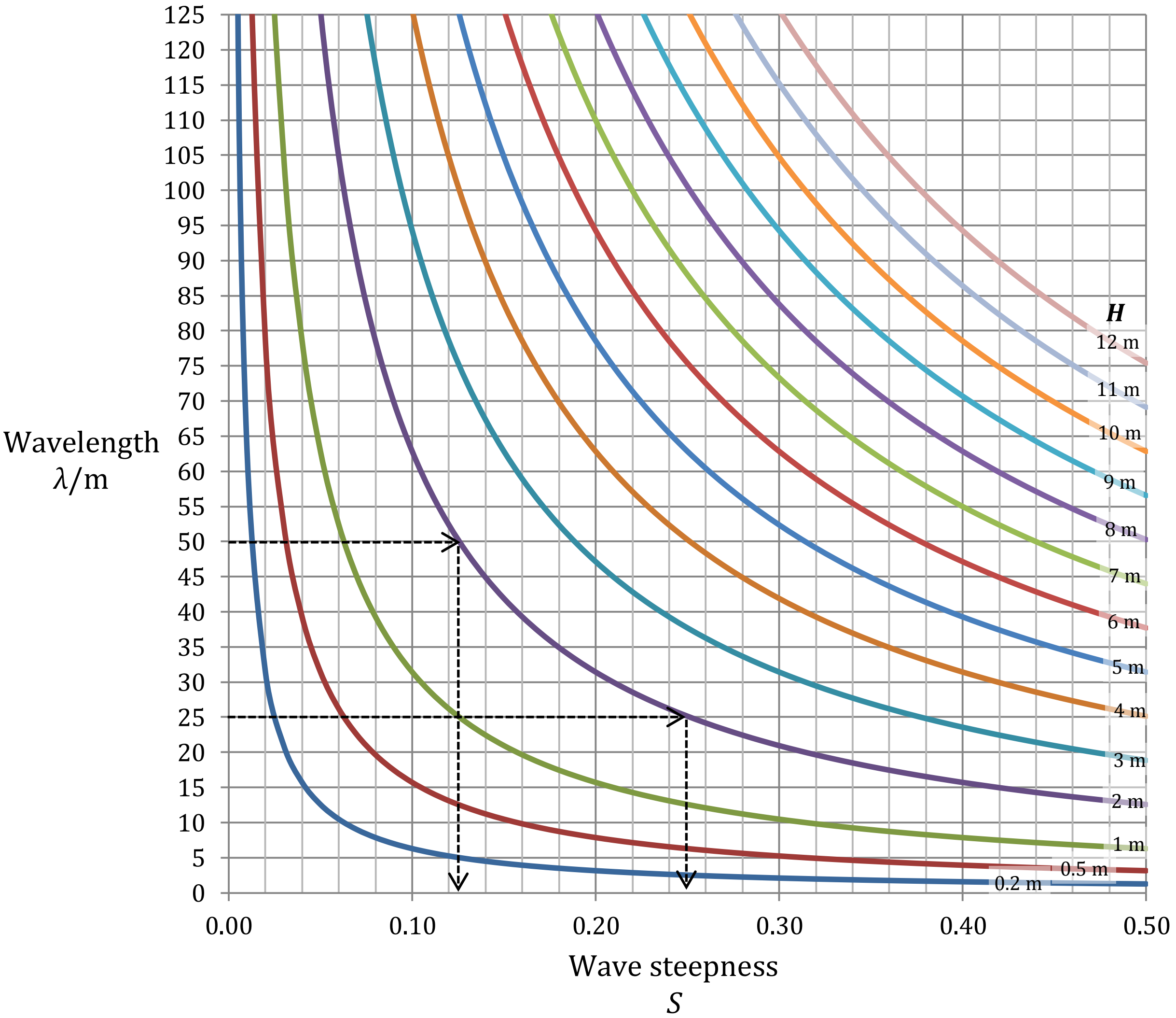
This chart is used in a similar way to chart 2, to find the wavelength of a wave of a given period *T* in water of depth *d*.

For example, if the wave period is 8 s the wavelength will be 42 m for depth *d* = 3 m, and 81 m for depth *d* = 15 m.

With the wavelength from chart 3 and the predicted wave height H, we can now use chart 4 to find the wave steepness.

Chart 4
Wavelength against steepness for different wave heights, *H*

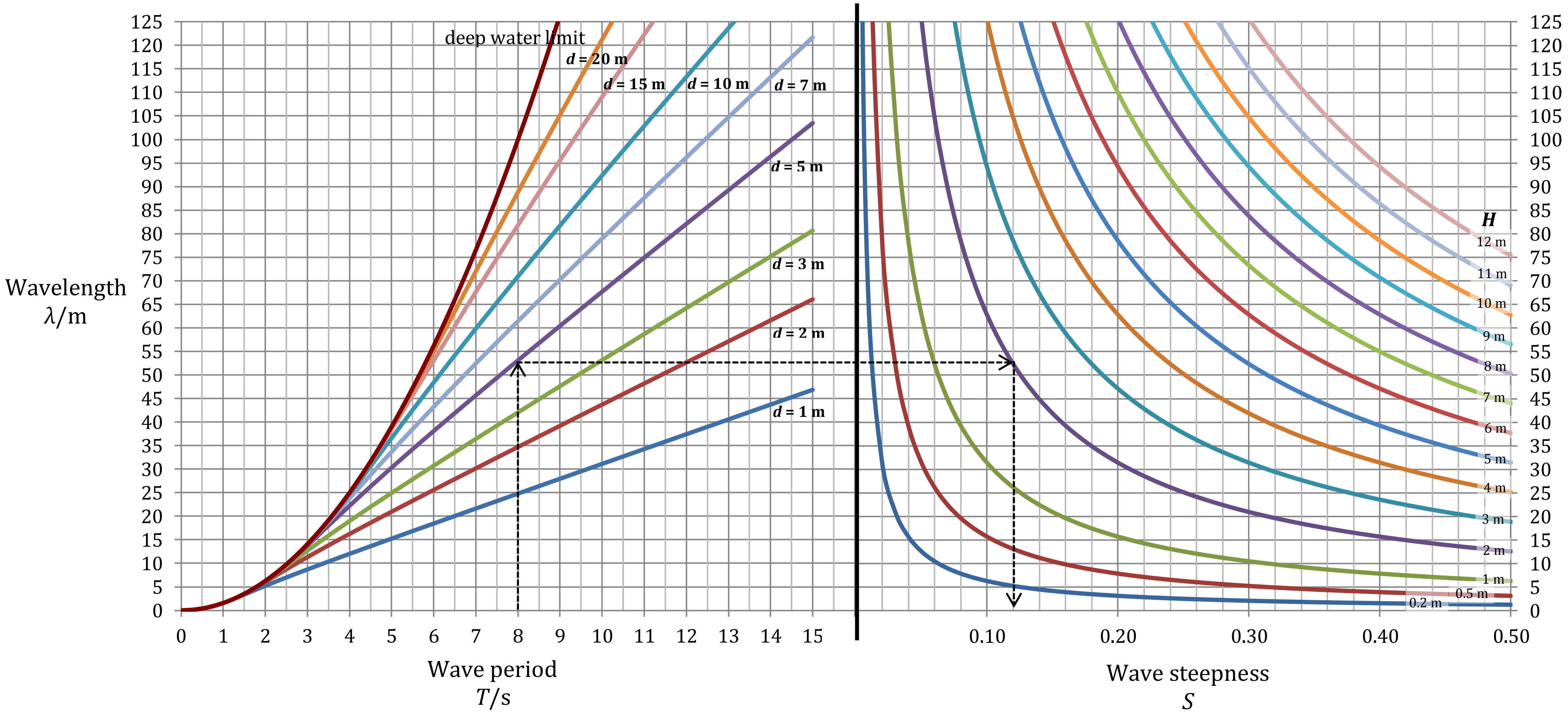
The lines on this chart are plotted from the equation $S = \frac{\pi H}{\lambda}$. Note that the chart is unconventional in that the independent variable λ is plotted on the y-axis, with the dependent variable S on the x-axis; the axes have been arranged in this way so that Chart 4 can be conveniently linked to the other charts to produce the combined charts below.



To use the chart, a horizontal line is drawn at the wavelength determined in chart 3, to meet the line corresponding to the wave height H. The steepness S is then read from the x-axis. For example a wave of wavelength 50 m and height H = 2.0 m has steepness S = 0.125. If the wavelength drops to 25 m but H remains the same, the steepness becomes S = 0.25.

Charts 3 and 4 can be combined to find the steepness *S* of waves with period *T* and height *H*, in water of depth *d*.

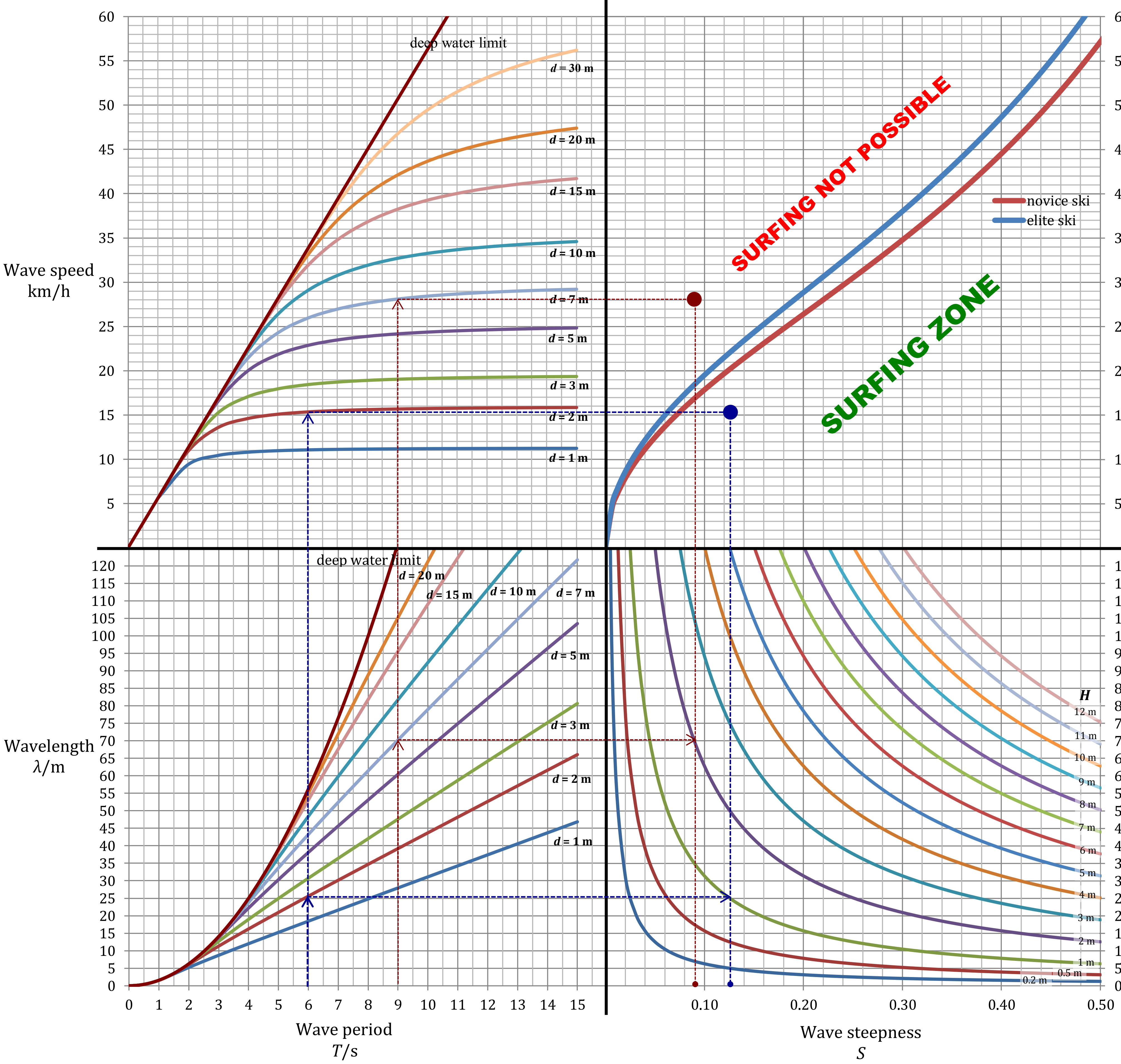
Charts 3 and 4 combined



For example, waves with period *T* = 8 s in water of depth *d* = 5 m, with wave height *H* = 2 m, have steepness *S* = 0.12

Master chart

Charts 1-4 can be further combined into a single 'master chart' containing all the information required to predict when effortless surfing is possible, and at what speed, given the wave period *T*, the wave height *H* and the water depth *d*.



To use the chart proceed as follows:

- Make a note of the expected period *T*, water depth *d*, and wave height *H* at the location where you plan to paddle (for example, *T* = 9 s , *d* = 7 m, *H* = 2 m).
- From the wave period axis (bottom left chart) draw a vertical line at your value of *T*, extending through the wavelength chart and into the wave speed chart.
- From the points where the *T* line cuts the curves corresponding to your value of *d* in the wavelength and the speed charts, draw horizontal lines. These lines indicate the wavelength and speed of the waves on their relevant axes.
- Where the wavelength line cuts the curve corresponding to your *H* value in the wave steepness chart, draw a vertical line to meet the wave speed line in top right chart. The location of the crossing point indicates if the waves will produce effortless surfing or not.

Two examples are shown on the chart:
The brown lines are for *T* = 9 s, *d* = 7 m, *H* = 2 m. The chart shows that these conditions correspond to waves with speed 28 km/h, wavelength 70 m and steepness 0.09. These conditions lie outside the surfing zone, and so will not produce effortless runs.

The dark blue lines are for *T* = 6 s, *d* = 2 m, *H* = 1 m. The charts show that these conditions correspond to waves with speed 15 km/h. wavelength 25 m and steepness 0.125. These waves can produce effortless surfing.

A printable copy of the chart is provided on page 22.

1.5 Applications

Together the four charts contain all the information needed to provide 'in-principle' answers to a wide range of questions about the surfability of waves. Some examples are as follows:

Q *Why can I surf effortlessly on the relatively small waves on a shallow lake/harbour with depth 1-2 m, but find I can't catch long runs on bigger ocean waves where the depth is >10 m?*

A The charts show that the maximum speed for long period waves on open water 1-2 m deep is 11-15 km/h. Short period waves (T< 4 s say) will be travelling more slowly than this. Waves travelling at 11 km/h can be surfed on a novice ski when S > 0.04. For T = 6 s this corresponds to a wave height of just over 20 cm; shorter period waves will be surfable at even lower wave heights. Waves travelling at 15 km/h can be surfed on a novice ski with S > 0.09, corresponding to a wave height of about 60 cm at T = 6 s.

In water depth d = 10 m waves with a period T = 6 s will be travelling at 29 km/h; longer period waves will be even faster. Surfing effortlessly at this speed requires S > 0.21 for an elite ski and S > 0.24 for a novice ski. With d = 10 m the corresponding wave heights are around H = 3.1 m and H =3.8 m - this is bigger than the conditions I venture out in, and so I will not be able to catch effortless runs at this speed.

Q *My ambition is to surf effortlessly at 20 km/h. What conditions will help me achieve this?*

A The charts show that to surf at 20 km/h I need to paddle in an area where the water depth is a little over 3 m. For my novice ski to surf at 20 km/h will require S > 0.13. To achieve this with minimum wave height I should choose a day when the wave period is as short as possible – let's say T = 5 s. For this combination of speed, depth and period a wave height of just over 1 m should produce an effortless run.

Q *What is the maximum possible speed for a surfski?*

A Let's assume that this is the maximum speed achieved by surfing in the direction of wave travel. The theoretical maximum steepness of a wave is S = 0.44. The charts show that for an elite ski, waves with this steepness are surfable up to a wave speed of 53.5 km/h. In deep water, waves travelling at this speed have wavelength 141 m. At this wavelength the wave height required to produce a steepness of 0.44 is H = 19.7 m (65 ft) - rather you than me!

In theory higher speeds could be achieved by paddling a longer ski with lower overall drag (a double ski for example) in order to catch faster moving waves; by traversing the face of the wave (as board surfers do), in which case the resultant speed is the vector sum of the wave velocity and the velocity of the ski across the wave face; or by surfing 'rogue' waves with S > 0.44. But please note: the assumptions made in the theory presented here are unlikely to be valid at these extreme speeds – see the discussion below.

1.6 Assumptions

The charts in this paper have been developed for an idealised model of the surfski/wave interaction.

The key assumptions are:

1. The wave profile is that of a trochoidal/Gerstner wave resulting from water particle motion in circular orbits.
2. The wave dispersion is given by $v_{\text{wave}} = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(2\pi\frac{d}{\lambda}\right)}$
3. The ski drag coefficient is independent of speed.
4. The motion is downwind with the wind speed equal to the wave speed.

Assumptions 1, 2 and 3 are approximations only.

I've used the trochoidal profile of Gerstner waves in this study because its equations are relatively simple. Mathematically, Gerstner waves are the one known exact solution for the equations that govern particle motion in a surface water wave; but physically they give only an approximation to the motion and wave shape (this is because Gerstner waves are not irrotational). Series solutions such as that by Stokes (3) are physically more accurate but much more complex. These show that the water particles have a small drift velocity in addition to their orbital velocity (the Stokes' drift). Although this will make some difference to the exact positions of the lines separating the surfing and non-surfing zones in Chart 1, the trends and general conclusions revealed by the charts should not be affected. I will leave it to someone else to refine the theory with more accurate wave representations.

The dispersion relation $v_{\text{wave}} = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(2\pi\frac{d}{\lambda}\right)}$ is accurate for low amplitude waves only ($S < 0.16$ say); as S increases the speed and wavelength values on the charts must be regarded as increasingly approximate.

Values used for the drag coefficients C_T of the elite and novice skis are based on the measured value for an ICF K1(5), and the relative performance of a K1 and a range of skis reported by Greg Barton of Epic kayaks (6). The overall drag of the elite ski is taken to be equal to that of the K1; the novice ski has 19% greater drag, which results in a speed difference of approximately 6% when paddled with the same power on flat water.

The K1 drag measurements extend to a speed of 18.6 km/h. C_T is approximately constant up to this speed (though starting to show some reduction at the higher speeds). It is likely that, on flat water, C_T will decrease further at speeds above 20 km/h as the hull begins to plane; however, on the slope of a wave, this reduction might be countered by increased drag due to bow burial or other factors. Without further experimental data the assumption of constant C_T is a reasonable starting point, and, as with the wave profile, small changes in C_T should not affect the trends and general conclusions revealed by the charts.

Assumption 4 - paddling downwind with wave speed equal to wind speed – simplifies the mathematics and is a realistic scenario. In practice when the wind speed exceeds the wave speed then surfing will be possible at lower wave steepness values and higher-speed waves can potentially be surfed; when the wind speed is less than the wave speed then effortless surfing will require steeper and/or slower waves.

1.7 Limitations and conclusions

To this point we have considered only the conditions for a wave to be surfable or not. We have not addressed the issue of 'catching' the wave in the first place. The ski will not surf spontaneously from rest – it will just bob up and down as the waves pass. To catch the wave, the ski must be travelling at some significant percentage of the wave speed as the wave arrives. The calculation of the required percentage will be the subject of the next paper in this series.

The theory here has been developed for waves of a given period and steepness. In reality the sea conditions on a given day are produced by the superposition of waves with a spectrum of periods and heights, travelling in a range of directions. The forecast wave period and height are average values, and characterise the 'typical' wave on that day, but there will be a range of waves with different speeds and heights around those average values.

The superposition of waves with a range of periods results in the formation of a *wave group*. This is a series of wave crests that appear to travel together as a unit. Closer observation reveals that the group-speed is lower than that of the individual crests, which progress in turn through the group from back to front.

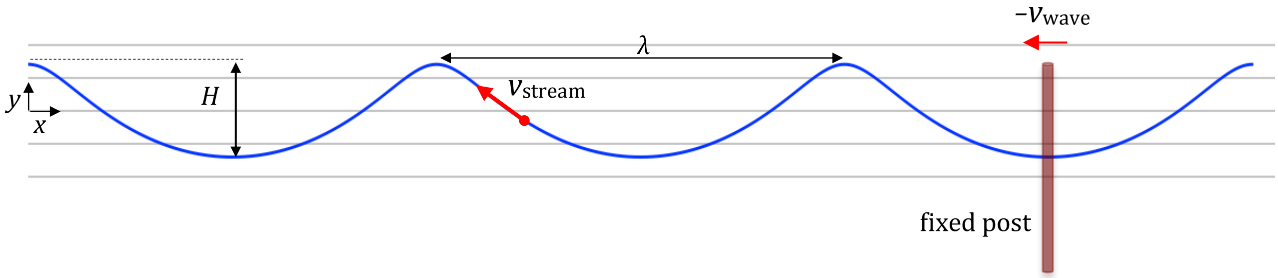
A wave crest rises at the back of a group, and then moves forwards through the group with increasing height until it becomes the peak of the group. As the crest continues through the group its height decays again until it disappears from the front of the group.

Since effortless surfing takes place on a wave crest, it will be at the velocity of the crest (the phase velocity); however, surfing can be sustained only for as long as the wave is of sufficient height. As the crest moves to the front of the group and decays, the ski will drop back onto the crest behind, and so on – the average speed the ski can maintain over a distance will thus be the group velocity, not the phase velocity. The physics of surfing wave groups will also be the subject of a future paper in this series.

PART 2
MATHEMATICS

2.1 Trochoidal waveform

Figure 2.1
Trochoidal waveform in the frame of reference in which the wave appears stationary



In the frame of reference in which the wave is stationary the waveform is given by the parametric equations:

$$x = \frac{\lambda}{2\pi}\theta - \frac{H}{2}\sin\theta$$

$$y = \frac{H}{2}\cos\theta$$

The location of a point on the wave surface is specified by the single parameter θ.
In this reference frame a post fixed in the sea floor appears to travel from right to left with velocity $-\boldsymbol{v}_{\text{wave}}$. The surface water particles are flowing from right to left along the surface of the wave with velocity $\boldsymbol{v}_{\text{stream}}$, which varies with position along the waveform; it is this velocity we need to find in order to make use of the surfability condition $\sin\gamma \geq \frac{C_T}{2gL} v_{\text{stream}}^2$.

2.2 Particle motion – calculation of v_{stream}^2

In the frame of reference of the fixed post, the surface water particles are moving in clockwise circular orbits with diameter H, period T and speed v_p given by:

$$v_p = \frac{\pi H}{T} = \frac{\pi H}{\lambda} v_{\text{wave}} = S v_{\text{wave}}$$

where S is the wave steepness.

Figure 2.2
Particle motion in the frame of reference of the fixed post

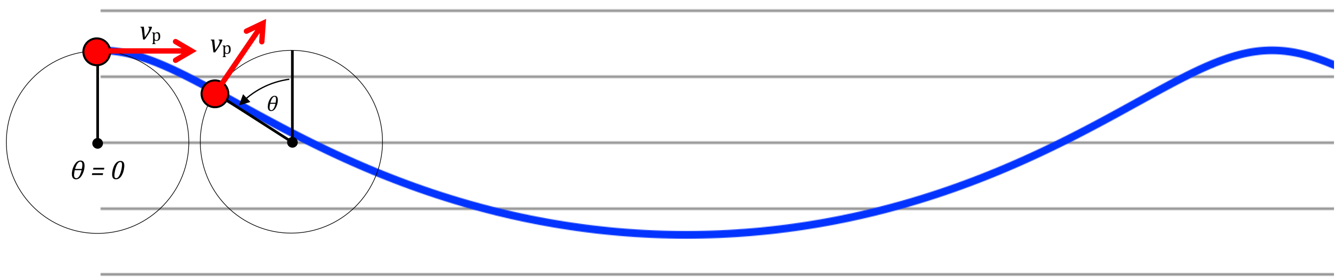


Figure 2.2 is plotted at time $t = 0$ when the origins of the two reference frames are coincident. The parameter θ used to locate a point on the waveform in Fig 2.1 can be identified as equal to the phase angle by which the motion of the particle at that point in Fig 2.2 lags that of the particle at $\theta = 0$.

The x and y components of a surface particle's velocity in this reference frame are given by:

$$v_{p_x} = v_p \cos\theta = S v_{\text{wave}} \cos\theta$$

$$v_{p_y} = v_p \sin\theta = S v_{\text{wave}} \sin\theta$$

We make the transformation to the stationary waveform frame by adding $-\boldsymbol{v}_{\text{wave}}$ to $\boldsymbol{v}_{\text{p}}$.

The components of $\boldsymbol{v}_{\text{stream}}$ are thus:

$$v_{\text{stream}_x} = Sv_{\text{wave}} \cos\theta - v_{\text{wave}} = v_{wave}(S\ \cos\theta - 1)$$

$$v_{\text{stream}_y} = v_{\text{wave}}\ S\sin\theta$$

From Pythagoras' theorem we have:

$$v_{\text{stream}}^2 = v_{\text{stream}_x}^2 + v_{\text{stream}_y}^2 = v_{\text{wave}}^2[(S\cos\theta - 1)^2 + S^2\sin^2\theta]$$

Which reduces to:

$$v_{\text{stream}}^2 = v_{\text{wave}}^2(1 - 2S\cos\theta + S^2)$$

The surfability condition thus becomes:

$$\sin\gamma \geq \frac{\mathrm{C_T}}{2gL} v_{\text{wave}}^2(1 - 2S\cos\theta + S^2)$$

2.3 Calculation of sin $\boldsymbol{\gamma}$

In the reference frame in which the wave is stationary, the surface particle velocity $\boldsymbol{v}_{\text{stream}}$ must be directed along the wave surface; any component perpendicular to the surface would cause the waveform to change shape, which it does not.

The inclination of $\boldsymbol{v}_{\text{stream}}$ to the horizontal is thus equal to that of the wave surface, which is the angle γ.

Figure 2.3 Relationship between $\boldsymbol{v}_{\text{stream}}$ and γ

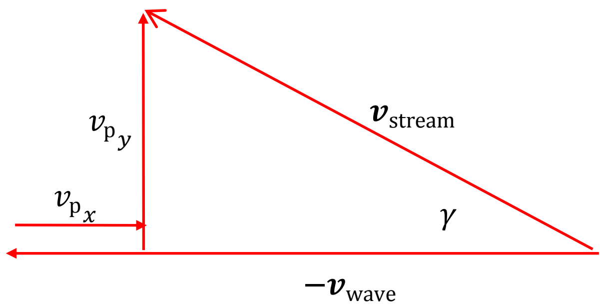


Figure 2.3 shows the vector composition of $\boldsymbol{v}_{\text{stream}}$. This demonstrates that:

$$\sin\gamma = \frac{v_{\text{p}_y}}{v_{\text{stream}}} = \frac{Sv_{\text{wave}}\sin\theta}{v_{\text{wave}}(1 - 2S\cos\theta + S^2)^{\frac{1}{2}}}$$

The surfability condition thus becomes:

$$\frac{S\sin\theta}{(1-2S\cos\theta+S^2)^{\frac{1}{2}}} \geq \frac{\mathrm{C_T}}{2\mathrm{g}L} v_{\mathrm{wave}}^2 (1-2S\cos\theta+S^2)$$

Finally, by rearranging to make v_{wave} the subject of the inequality, we obtain the surfability condition for trochoidal waves of steepness S and speed v_{wave}:

$$v_{\mathrm{wave}} \leq \sqrt{\frac{2\mathrm{g}L}{\mathrm{C_T}}} \sqrt{\frac{S\sin\theta}{(1-2S\cos\theta+S^2)^{3/2}}}$$

2.4 $\boldsymbol{\theta_{\max}}$ – the surfing 'sweet spot'

The two lines that divide Chart 1 into surfing and non-surfing zones are plots of the equation:

$$v_{\mathrm{wave}} = \sqrt{\frac{2\mathrm{g}L}{\mathrm{C_T}}} \sqrt{\frac{S\sin\theta_{max}}{(1-2S\cos\theta_{max}+S^2)^{3/2}}}$$

for two values of $\sqrt{\frac{2\mathrm{g}L}{\mathrm{C_T}}}$. In this equation $\theta_{\max}$ is the value of θ that maximises $\frac{S\sin\theta}{(1-2S\cos\theta+S^2)^{3/2}}$.

With the aid of Wolfram Mathematica we locate $\theta_{\max}$ at:

$$\theta_{\max} = \cos^{-1}\left(\frac{-S^2+\sqrt{S^4+14S^2+1}-1}{2S}\right)$$

The parameter $\theta_{\max}$ identifies the location of the point on the waveform that is most readily surfed – the surfing 'sweet spot'. The *x* coordinate of this point (its horizontal distance in front of the wave crest) is given by:

$$x = \frac{\lambda}{2\pi}\theta_{\max} - \frac{H}{2}\sin\theta_{\max}$$

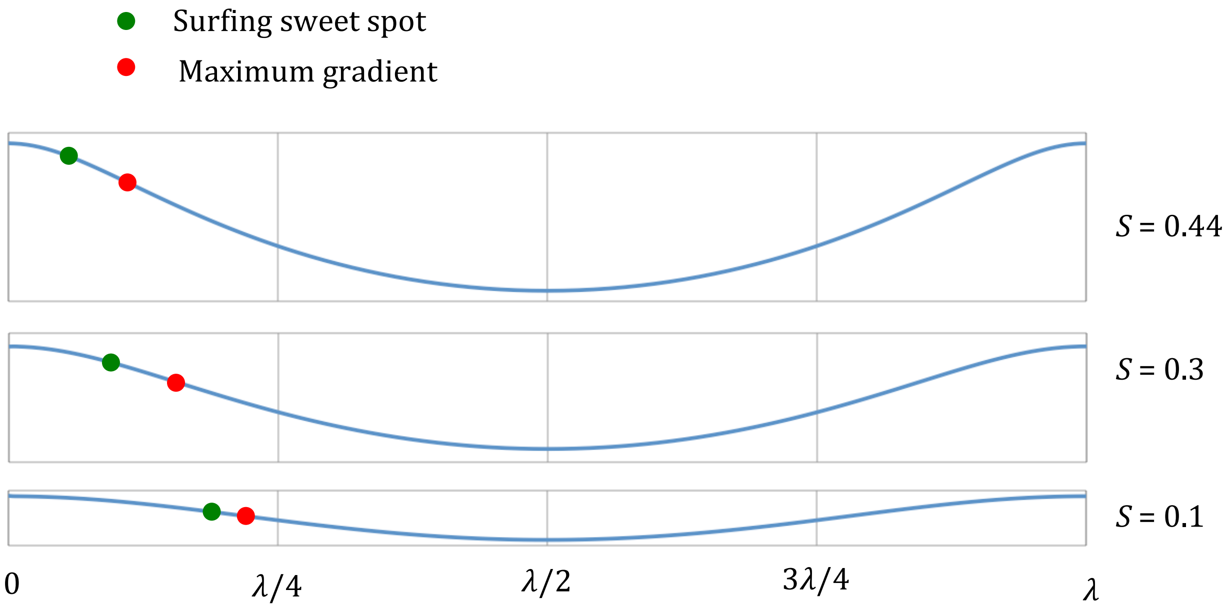
The green dots and line in Figure 2.4 shows how, for a trochoidal wave, the position of the sweet spot moves progressively further up the wave crest as the wave steepness increases.

It's interesting to note that the sweet spot does not coincide with the point of maximum gradient of the waveform (maximum γ). The location of maximum gradient is shown by the red dots and line in Figure 2.4. For the trochoidal curve the location of maximum γ is given by:

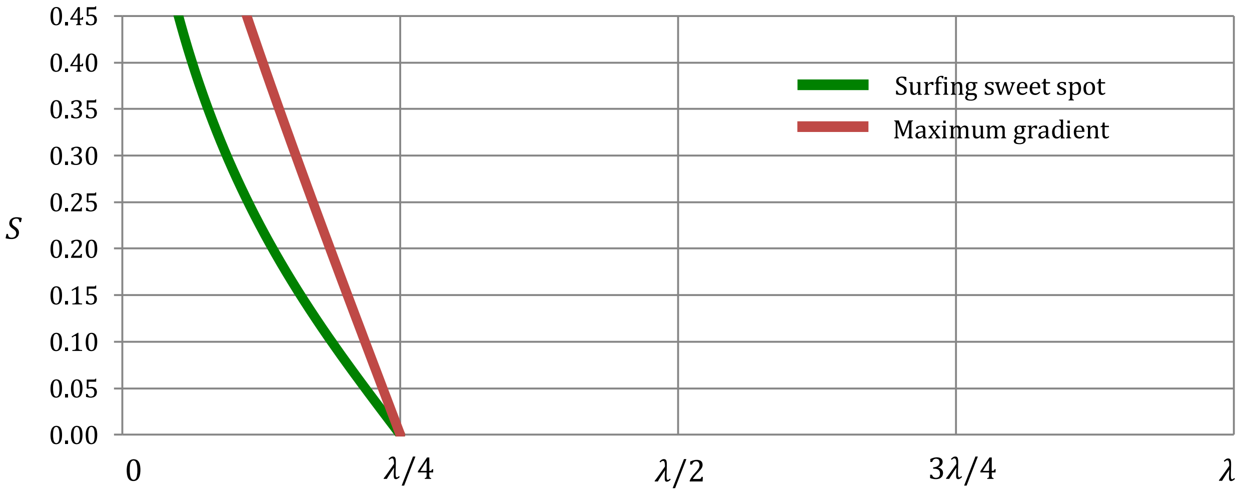
$$\theta_{\gamma_{max}} = \cos^{-1} S$$

Figure 2.4
Locations of the surfing sweet spot and maximum wave gradient (γ_{max}) for trochoidol waves as a function of steepness

(a) Locations on the waveform for three steepness values



(b) Plots showing how the locations of the surfing sweet spot and the maximum gradient move towards the wave crest as the wave steepness S increases.



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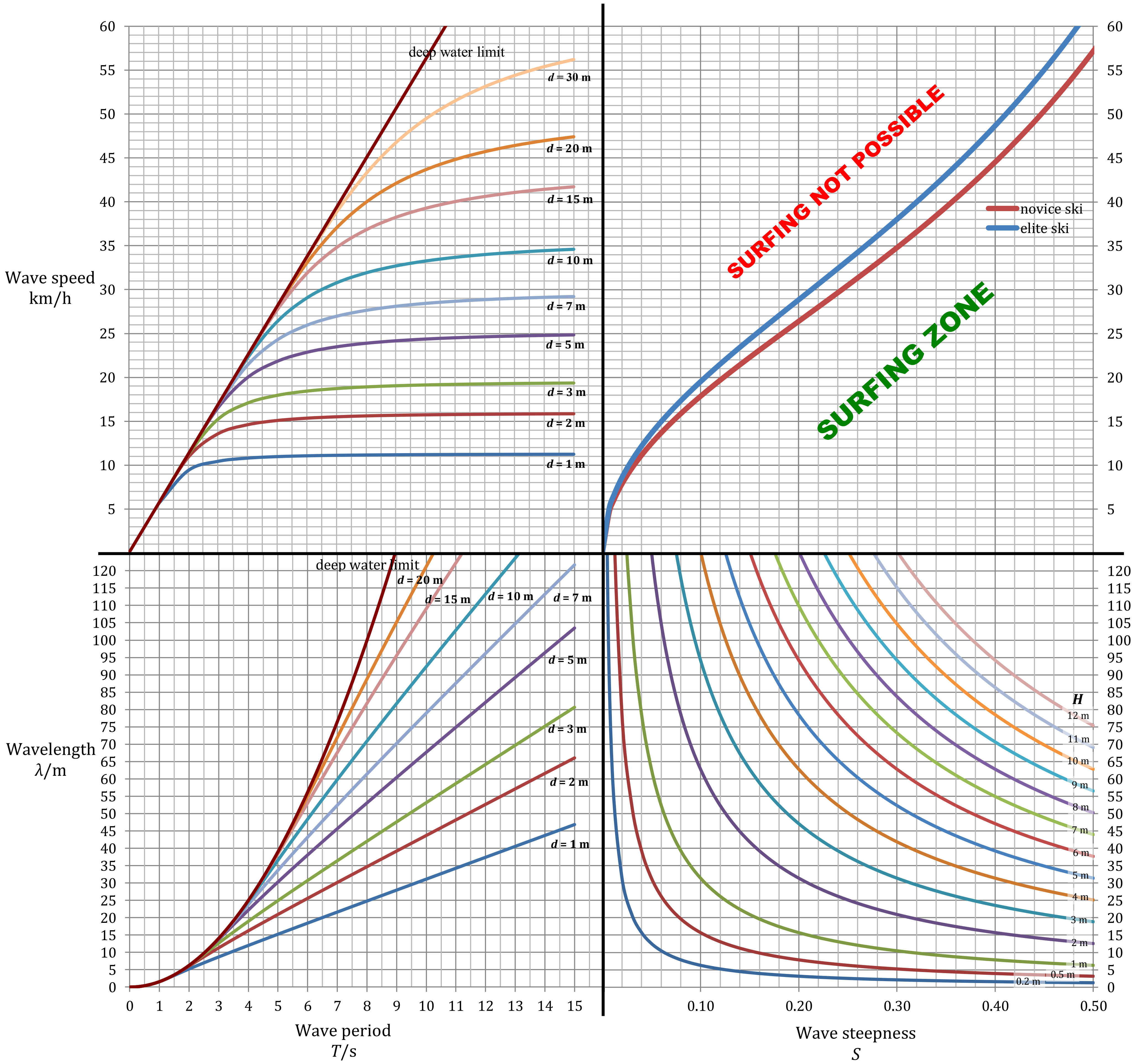
SURFSKI WAVE 'SURFABILITY' CHART

This chart contains all the information required to predict when effortless surfing is possible given the wave period *T*, the wave height *H* and the water depth *d.*



To use the chart proceed as follows:

- Make a note of the expected period *T*, water depth *d*, and wave height *H* at the location where you plan to paddle (for example, *T* = 9 s , *d* = 7 m, *H* = 2 m).
- From the wave period axis (bottom left chart) draw a vertical line at your value of *T,* extending through the wavelength chart and into the wave speed chart.
- From the points where the *T* line cuts the curves corresponding to your value of *d* in the wavelength and the speed charts, draw horizontal lines. These lines indicate the wavelength and speed of the waves on their relevant axes.
- Where the wavelength line cuts the curve corresponding to your *H* value in the wave steepness chart, draw a vertical line to meet the wave speed line in top right chart. The location of the crossing point indicates if the waves will produce effortless surfing or not.